Theorem: f piecewise smooth

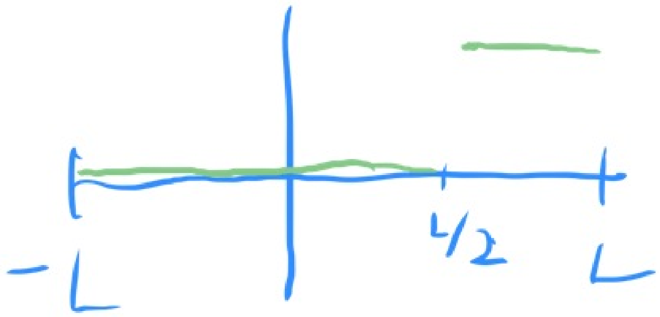
$$\tilde{f}(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{L}x + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x$$

its Fourier series

$$\Rightarrow \quad \tilde{f}(x) = \begin{cases} f(x) & \text{if } f \text{ continuous at } x \\ \frac{1}{2}\left[f(x-) + f(x+)\right] & \text{at jumps} \end{cases}$$

Example:

$$f(x) = \begin{cases} 1 & L/2 \le x \le L \\ 0 & -L \le x < L/2 \end{cases}$$



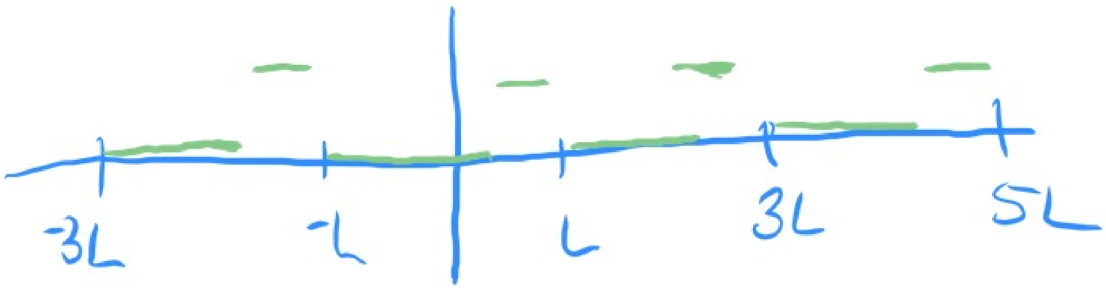
$y = f(x)$

$y = \tilde{f}(x)$

jump at $L/2$

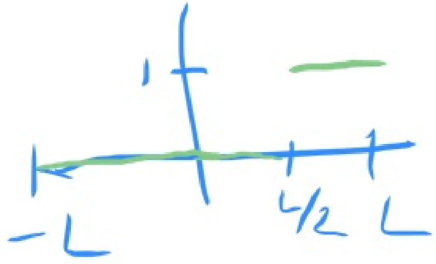
and jumps at $\pm L$

can be visualized by looking at its periodic extension



have jumps at $\pm L$

Let's calculate Fourier series for f



$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$= \frac{1}{2L}\int_{L/2}^{L} 1 \cdot dx$$

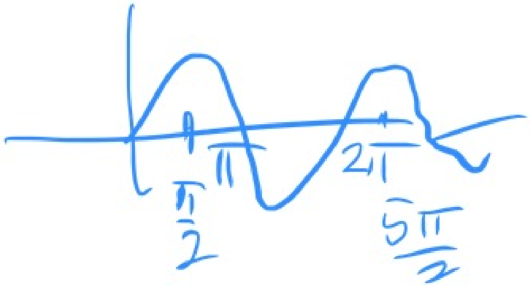
$$= \frac{1}{2L}(L/2) = \boxed{\frac{1}{4}}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}x\,dx$$

$$= \frac{1}{L}\int_{L/2}^{L} \cos\frac{n\pi}{L}x\,dx$$

$$= \frac{1}{L} \frac{L}{n\pi} \sin \frac{n\pi}{L} x \Big|_{L/2}^{L}$$

$$= \frac{1}{n\pi} \left(\sin n\pi - \sin \frac{n\pi}{2} \right)$$



$\sin n\pi = 0 \quad \forall n$

$$\sin \frac{n\pi}{2} = \begin{cases} 0 & n \text{ even} \\ 1 & n = 4m+1 \\ -1 & n = 4m+3 \end{cases}, \quad m = 0, 1 \ldots$$

$$= \frac{1}{n\pi} \left(-\sin \frac{n\pi}{2} \right)$$

$a_0 = 1/4$

cosine part.

$$a_n = \begin{cases} 0 & n \text{ even} \\ -\frac{1}{n\pi} & n = 4m+1 \\ \frac{1}{n\pi} & n = 4m+3 \end{cases}$$

Get $\tilde{f}(x) = \frac{1}{4} + \sum_{n \text{ odd}} (-1)^{(n+1)/2} \frac{1}{n\pi} \cos \frac{n\pi}{L} x$

$$+ \sum b_n \sin \frac{n\pi}{L} x$$

<u>Moral:</u> Fourier theorem allows us to calculate $\tilde{f}(x)$ with hardly any additional work.

<u>Remark:</u> Fourier theorem can be used to calculate certain infinite series

eg. $$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

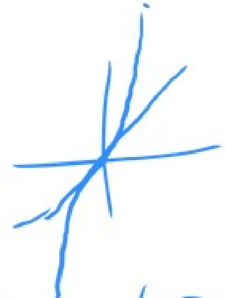
5.3 Sine and cosine series

Def. A function $f(x)$ is called **odd** if

$$f(-x) = -f(x)$$

examples: $f(x) = x, \; x^3, \; x^5 \ldots$

$$f(x) = \sin \frac{n\pi}{L} x$$

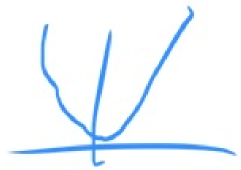


A function $f(x)$ is called **even** if

$$f(-x) = f(x)$$

examples: $f(x) = \text{const}, \; x^2, \; x^4 \ldots$

$$f(x) = \cos \frac{n\pi}{L} x$$



Obs. f odd, g even $\Rightarrow fg = \text{odd}$

f odd, g odd $\Rightarrow fg = \text{even}$

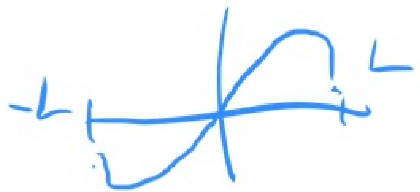
etc.

<u>Observation:</u> f odd $\Rightarrow$ all cosine Fourier coeff are equal to 0

f even $\Rightarrow$ all sine Fourier coeff. are 0

For cosine coeff:

$$a_n = \frac{1}{L} \int_{-L}^{L} \underbrace{\underset{\text{odd}}{f(x)} \, \underset{\text{even}}{\cos \frac{n\pi}{L} x}}_{\text{odd}} \, dx$$



reason: integral $\int_{-L}^{L} g(x)dx = 0$ for any odd function.

apply this for $g(x) = f(x) \cos \frac{n\pi}{L} x$.

Odd and even extensions:

now: $f : [0, L] \to \mathbb{R}$

have both a sine Fourier series and a cosine Fourier series

sine series: $\tilde{f} = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{L} x$

where $a_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x \, dx$

Theorem Both sine series and cosine series approximate f as in previous theorem (for jumps, get average etc)

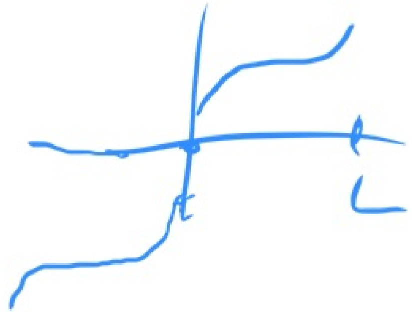
proof of this can be reduced to the one of previous theorems for $[-L, L]$.

Observe: can extend f

to an odd function $f^{odd}\ [-L,L] \to \mathbb{R}$

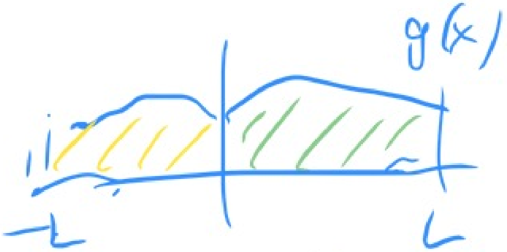
by $f^{odd}(-x) = -f(x)$

(reflect at origin



$\Rightarrow$ sine coeff of f^{odd} : $[-L,L)$

$$= \frac{1}{L}\int_{-L}^{L} \underbrace{\underbrace{f^{odd}(x)}_{odd}\ \underbrace{\sin\frac{n\pi}{L}x}_{odd}}_{even}\, dx$$



green area = yellow area

for any even fct g

$$\int_{-L}^{L} g(x)\,dx = 2\int_{0}^{L} g(x)\,dx$$

$$\| \quad \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi}{L}x\,dx = \text{Fourier coeff of sine serie}$$

Moral: Can reduce questions about convergence of sine series on $[0,L]$ for a function f to the same question for f^{odd} on $[-L,L]$.

- same Fourier coefficients (for sine series)
- same values at $[0,L]$